Supplementary Materials: Understanding fiber-optic sensitivity to a wavefield A framework to separate site amplification from orientation effects

1 Linear fiber sensitivity equations and demonstration

In these demonstrations, we consider a plane waves propagating in direction $\vec{K}$ as described in equation 1, with φ_r and θ_r the azimuth and the dip of the ray, both defined to progress in trigonometric direction as depicted in Figure S1.

$$\begin{aligned}\vec{K} &= [k_x, k_y, k_z] \\ &= k[\cos(\varphi_r)\cos(\theta_r), \sin(\varphi_r)\cos(\theta_r), \sin(\theta_r)]\end{aligned} \tag{1}$$

We begin with a set of polarization vectors that describe the displacement for each wave type. Taking the time derivative yields particle velocity. Then applying a spatial derivative using the small strain tensor, gives strain rate. The gauge length effect is accounted for by averaging the strain rate over a linear segment aligned with the fiber direction. The fiber orientation is described by φ_D its azimuth and θ_D its dip, both also defined to progress in trigonometric direction, as can be seen on Figure S1.

1.0.1 Abbreviation Summary

- A: the displacement amplitude
- $\psi(x, y, z, t)$: The oscillatory factor for P- and S-waves, expressed as

$$e^{ik(ct - x\cos(\varphi_r)\cos(\theta_r) - y\sin(\varphi_r)\cos(\theta_r) - z\sin(\theta_r))}$$

- $\dot{\epsilon}_{xx'}$: The strain rate in the xx' direction, rotated into the fiber's direction
- $\dot{\epsilon}_{vl}$: The strain rate in a vertical loop
- $\dot{\epsilon}_{hl}$: The strain rate in a horizontal loop
- φ_r: The ray azimuth
- θ_r: The ray dip
- φ_D: The DAS azimuth
- θ_D: The DAS dip
- R: The gauge length loop radius
- g: The gauge length

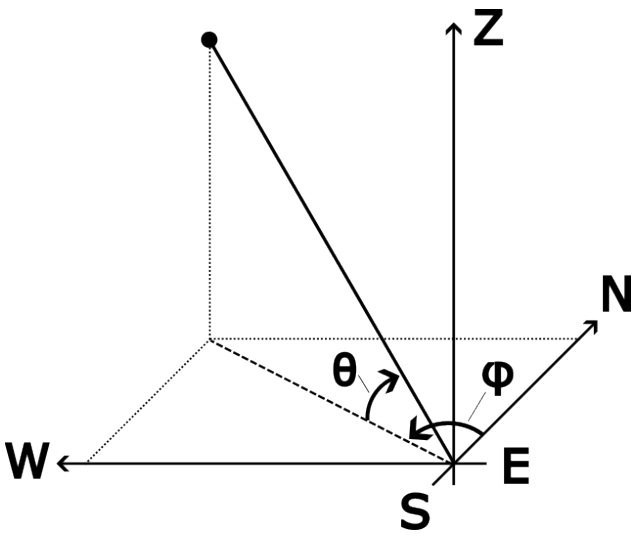


Figure S1: Spherical coordinate system used in this study, where the azimuth φ is defined in the horizontal plane, measured from north and increasing counterclockwise (trigonometric convention), corresponding to positive values toward the west. The dip θ is likewise defined following the trigonometric convention, with positive values indicating upward directions relative to the horizontal plane.

1.1 Rotational matrix and strain tensor in xx' direction

Creation of a rotation matrix for the xx component of the strain tensor for all azimuth (x-axis rotation), and all dip (y-axis rotation)

$$\begin{bmatrix} \cos(\theta_D) & 0 & \sin(\theta_D) \\ 0 & 1 & 0 \\ -\sin(\theta_D) & 0 & \cos(\theta_D) \end{bmatrix} \begin{bmatrix} \cos(\varphi_D) & \sin(\varphi_D) & 0 \\ -\sin(\varphi_D) & \cos(\varphi_D) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) & \sin(\varphi_D)\cos(\theta_D) & \sin(\theta_D) \\ -\sin(\varphi_D) & \cos(\varphi_D) & 0 \\ -\sin(\theta_D)\cos(\varphi_D) & \cos(\theta_D) & \cos(\theta_D) \end{bmatrix}$$

Rotation of the ϵ_{xx} component of the strain tensor:

$$M \cdot \epsilon \cdot M^T$$

$$\epsilon_{xx'} = \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) & \sin(\varphi_D)\cos(\theta_D) & \sin(\theta_D) \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \cdot \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) \\ \sin(\varphi_D)\cos(\theta_D) \\ \sin(\theta_D) \end{bmatrix}$$

using the conservation of angular momentum ($\epsilon_{ij} = \epsilon_{ji}$)

$$\begin{aligned} \epsilon_{xx'} = &\cos(\varphi_D)^2 \cos(\theta_D)^2 \epsilon_{xx} + \sin(2\varphi_D)\cos(\theta_D)^2 \epsilon_{yx} + \sin(2\theta_D)\cos(\varphi_D)\epsilon_{xz} \\ &+ \sin(\varphi_D)^2 \cos(\theta_D)^2 \epsilon_{yy} + \sin(2\theta_D)\sin(\varphi_D)\epsilon_{zy} + \sin(\theta_D)^2 \epsilon_{zz} \end{aligned} \tag{2}$$

1.2 P waves

Starting from the P wave polarization displacement vector $\vec{u}$

$$\vec{u} = \begin{bmatrix} A\cos(\varphi_r)\cos(\theta_r)\psi(x,y,z,t) \\ A\sin(\varphi_r)\cos(\theta_r)\psi(x,y,z,t) \\ A\sin(\theta_r)\psi(x,y,z,t) \end{bmatrix}$$

$$
\begin{aligned}
u_{x'} &= \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) & \sin(\varphi_D)\cos(\theta_D) & \sin(\theta_D) \\ -\sin(\varphi_D) & \cos(\varphi_D) & 0 \\ -\sin(\theta_D)\cos(\varphi_D) & \cos(\theta_D) & \cos(\theta_D) \end{bmatrix} \begin{bmatrix} A\cos(\varphi_r)\cos(\theta_r)\,\psi(x,y,z,t) \\ A\sin(\varphi_r)\cos(\theta_r)\,\psi(x,y,z,t) \\ A\sin(\theta_r)\,\psi(x,y,z,t) \end{bmatrix} \\
&= \boxed{A\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_r-\varphi_D)+\sin(\theta_r)\sin(\theta_D)\right]\psi(x,y,z,t)}
\end{aligned}
$$

We apply a time derivative to get the velocity vector $\dot{\vec{u}}$

$$
\dot{\vec{u}} = \begin{bmatrix} Akic\cos(\varphi_r)\cos(\theta_r)\,\psi(x,y,z,t) \\ Akic\sin(\varphi_r)\cos(\theta_r)\,\psi(x,y,z,t) \\ Akic\sin(\theta_r)\,\psi(x,y,z,t) \end{bmatrix}
$$

$$
\dot{u}_{x'} = \frac{\delta u_{x'}}{\delta t} = \boxed{Aik\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_r-\varphi_D)+\sin(\theta_r)\sin(\theta_D)\right]\psi(x,y,z,t)}
$$

The strain-rate tensor component in the ij-direction is obtained by taking the spatial gradient, giving $\dot{\epsilon}_{ij}$

$$
\begin{aligned}
\dot{\epsilon}_{xx} &= k^2cA\cos(\varphi_r)^2\cos(\theta_r)^2\,\psi(x,y,z,t) \\
\dot{\epsilon}_{xy} &= \tfrac{k^2cA}{2}\sin(2\varphi_r)\,\psi(x,y,z,t) \\
\dot{\epsilon}_{xz} &= \tfrac{k^2cA}{2}\cos(\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) \\
\dot{\epsilon}_{yx} &= \tfrac{k^2cA}{2}\sin(2\varphi_r)\cos(\theta_r)^2\,\psi(x,y,z,t) \\
\dot{\epsilon}_{yy} &= k^2cA\sin(\varphi_r)^2\cos(\theta_r)^2\,\psi(x,y,z,t)
\end{aligned}
\qquad
\begin{aligned}
\dot{\epsilon}_{yz} &= \tfrac{k^2cA}{2}\sin(\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) \\
\dot{\epsilon}_{zx} &= \tfrac{k^2cA}{2}\cos(\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) \\
\dot{\epsilon}_{zy} &= \tfrac{k^2cA}{2}\sin(\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) \\
\dot{\epsilon}_{zz} &= k^2cA\sin(\theta_r)^2\,\psi(x,y,z,t)
\end{aligned}
$$

Substituting in equation 2 we get the strain rate in direction xx'

$$
\begin{aligned}
\dot{\epsilon}^P_{xx'} =& k^2cA\Big[\cos(\varphi_D)^2\cos(\theta_D)^2\cos(\varphi_r)^2\cos(\theta_r)^2+\cos(\varphi_D)\sin(\varphi_D)\cos(\theta_D)^2\sin(2\varphi_r)\cos(\theta_r)^2 \\
&+\cos(\varphi_D)\cos(\theta_D)\sin(\theta_D)\cos(\varphi_r)\sin(2\theta_r)+\sin(\varphi_D)^2\cos(\theta_D)^2\sin(\varphi_r)^2\cos(\theta_r)^2 \\
&+\sin(\varphi_D)\sin(\theta_D)\cos(\theta_D)\sin(\varphi_r)\sin(2\theta_r)+\sin(\theta_D)^2\sin(\theta_r)^2\Big]\,psi(x,y,z,t) \\
=& k^2cA\Big[\cos(\theta_D)^2\cos(\theta_r)^2\cos(\varphi_D-\varphi_r)^2+2\sin(\theta_r)\cos(\theta_r)\cos(\theta_D)\sin(\theta_D)\cos(\varphi_D-\varphi_r) \\
&+\sin(\theta_D)^2\sin(\theta_r)^2\Big]\,\psi(x,y,z,t) \\
=& \boxed{k^2cA\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\right]^2\psi(x,y,z,t)}
\end{aligned}
$$

To consider the gauge length effect, we average the strain over a linear segment of direction

$[\cos(\varphi_D)\cos(\theta_D), \sin(\varphi_D)\cos(\theta_D), \sin(\theta_D)]$ and length g.

$$
\begin{aligned}
\dot{\epsilon}^P_{xx'g} =& \frac{1}{g}\int_{\frac{+g}{2}}^{\frac{-g}{2}} \dot{\epsilon}^P_{xx'}(x + v\cos(\varphi_D)\cos(\theta_D), y + v\sin(\varphi_D)\cos(\theta_D), z + v\sin(\theta_D), t)dv \\
=& \frac{k^2cA}{g}\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right]^2 \int_{\frac{+g}{2}}^{\frac{-g}{2}} e^{ik(ct - x\cos(\varphi_r)\cos(\theta_r) - y\sin(\varphi_r)\cos(\theta_r) - z\sin(\theta_r))} \\
& \times e^{-ik(v\cos(\varphi_D)\cos(\theta_D)\cos(\varphi_r)\cos(\theta_r) - v\sin(\varphi_D)\cos(\theta_D)\sin(\varphi_r)\cos(\theta_r) - v\sin(\theta_D)\sin(\theta_r))}dv \\
=& \frac{k^2cA}{g}\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right]^2 \\
& \times \left[\frac{e^{-ikv(\cos(\varphi_D)\cos(\theta_D)\cos(\varphi_r)\cos(\theta_r) + \sin(\varphi_D)\cos(\theta_D)\sin(\varphi_r)\cos(\theta_r) + \sin(\theta_D)\sin(\theta_r))}}{ik(\cos(\varphi_D)\cos(\theta_D)\cos(\varphi_r)\cos(\theta_r) + \sin(\varphi_D)\cos(\theta_D)\sin(\varphi_r)\cos(\theta_r) + \sin(\theta_D)\sin(\theta_r))}\right]_{\frac{+g}{2}}^{\frac{-g}{2}} \psi(x,y,z,t) \\
=& \frac{kcA}{g}\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right]\left[\frac{e^{-ikv((\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r))}}{i}\right]_{\frac{+g}{2}}^{\frac{-g}{2}} \psi(x,y,z,t) \\
=& \boxed{\begin{aligned}&\frac{2kcA}{g}\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right] \\ &\times \sin\left(\frac{kg}{2}\left(\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right)\right)\psi(x,y,z,t)\end{aligned}}
\end{aligned}
\tag{3}
$$

1.3 SH waves

Starting from the SH wave polarization displacement vector $\vec{u}$

$$\vec{u} = \begin{bmatrix} A\sin(\varphi_r)\psi(x,y,z,t) \\ A\cos(\varphi_r)\psi(x,y,z,t) \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
u_{x'} =& \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) & \sin(\varphi_D)\cos(\theta_D) & \sin(\theta_D) \\ -\sin(\varphi_D) & \cos(\varphi_D) & 0 \\ -\sin(\theta_D)\cos(\varphi_D) & \cos(\theta_D) & \cos(\theta_D) \end{bmatrix} \begin{bmatrix} A\sin(\varphi_r)\psi(x,y,z,t) \\ A\cos(\varphi_r)\psi(x,y,z,t) \\ 0 \end{bmatrix} \\
=& \boxed{A\left[\cos(\theta_D)\sin(\varphi_r - \varphi_D)\right]\psi(x,y,z,t)}
\end{aligned}
$$

We apply a time derivative to get the velocity vector $\dot{\vec{u}}$

$$\dot{\vec{u}} = \begin{bmatrix} Akic\sin(\varphi_r)\psi(x,y,z,t) \\ Akic\cos(\varphi_r)\psi(x,y,z,t) \\ 0 \end{bmatrix}$$

$$\dot{u}_{x'} = \frac{\delta\dot{u}}{\delta t} = \boxed{Akic\left[\cos(\theta_D)\sin(\varphi_r - \varphi_D)\right]\psi(x,y,z,t)}$$

The strain-rate tensor component in the ij-direction is obtained by taking the spatial gradient, giving $\dot{\epsilon}_{ij}$

$$
\begin{aligned}
\dot{\epsilon}_{xx} =& \frac{k^2cA}{2}\sin(2\varphi_r)\cos(\theta_r)\psi(x,y,z,t) & & \\
\dot{\epsilon}_{xy} =& k^2cA\sin(\varphi_r)^2\cos(\theta_r)\psi(x,y,z,t) & \dot{\epsilon}_{yz} =& -k^2cA\cos(\varphi_r)\sin(\theta_r)\psi(x,y,z,t) \\
\dot{\epsilon}_{xz} =& k^2cA\sin(\varphi_r)\sin(\theta_r)\psi(x,y,z,t) & \dot{\epsilon}_{zx} =& 0 \\
\dot{\epsilon}_{yx} =& -k^2cA\cos(\varphi_r)^2\cos(\theta_r)\psi(x,y,z,t) & \dot{\epsilon}_{zy} =& 0 \\
\dot{\epsilon}_{yy} =& \frac{-k^2cA}{2}\sin(2\varphi_r)\cos(\theta_r)\psi(x,y,z,t) & \dot{\epsilon}_{zz} =& 0
\end{aligned}
$$

Substituting in equation 2 we get the strain rate in direction xx'

$$\begin{aligned}
\dot{\epsilon}_{xx'}^{SH} =& ck^2 A \left[\frac{\cos(\varphi_D)^2 \cos(\theta_D)^2 \sin(2\varphi_r)\cos(\theta_r)}{2} + \frac{\sin(2\varphi_D)\cos(\theta_D)^2\cos(\theta_r)(\sin(\varphi_r)^2 - \cos(\varphi_r)^2)}{2} \right. \\
&+ \frac{\sin(2\theta_D)\cos(\varphi_D)\sin(\varphi_r)\sin(\theta_r)}{2} - \frac{\sin(\varphi_D)^2\cos(\theta_D)^2\sin(2\varphi_r)\cos(\theta_r)}{2} \\
&\left. - \frac{\sin(2\theta_D)\sin(\varphi_D)\cos(\varphi_r)\sin(\theta_r)}{2} \right] \psi(x,y,z,t) \\
=& \frac{ck^2A}{2} \left[\sin(2\varphi_r)\cos(\theta_r)\cos(\theta_D)^2\cos(2\varphi_D) - \sin(2\varphi_D)\cos(\theta_D)^2\cos(\theta_r)\cos(2\varphi_r) \right. \\
&\left. + \sin(2\theta_D)\sin(\theta_r)\sin(\varphi_r - \varphi_D) \right] \psi(x,y,z,t) \\
=& \boxed{ck^2 A \sin(\varphi_r - \varphi_D)\cos(\theta_D)\left[\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r - \varphi_D) + \sin(\theta_D)\sin(\theta_r)\right]\psi(x,y,z,t)}
\end{aligned}$$

To consider the gauge length effect, we average the strain over a linear segment of direction $[\cos(\varphi_D)\cos(\theta_D), \sin(\varphi_D)\cos(\theta_D), \sin(\theta_D)]$ and length g

$$\begin{aligned}
\dot{\epsilon}_{xx'g}^{SH} =& \frac{1}{g}\int_{\frac{+g}{2}}^{\frac{-g}{2}} \dot{\epsilon}_{xx'}^{SH}(x + v\cos(\varphi_D)\cos(\theta_D), y + v\sin(\varphi_D)\cos(\theta_D), z + v\sin(\theta_D), t)dv \\
=& \frac{k^2cA}{g}\sin(\varphi_r - \varphi_D)\cos(\theta_D)\left[\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r - \varphi_D) + \sin(\theta_D)\sin(\theta_r)\right] \\
&\times \int_{\frac{+g}{2}}^{\frac{-g}{2}} e^{ik(ct - (x + v\cos(\varphi_D)\cos(\theta_D))\cos(\varphi_r)\cos(\theta_r) - (y + v\sin(\varphi_D)\cos(\theta_D))\sin(\varphi_r)\cos(\theta_r) - (z + v\sin(\theta_D))\sin(\theta_r)} dv \\
=& \frac{ck^2A}{kg}\sin(\varphi_r - \varphi_D)\cos(\theta_D)\left[\frac{\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r - \varphi_D) + \sin(\theta_D)\sin(\theta_r)}{\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r - \varphi_D) + \sin(\theta_D)\sin(\theta_r)}\right] \\
&\times sin\left(\frac{kg}{2}\left(\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right)\right)\psi(x,y,z,t) \\
=& \boxed{\frac{2ckA}{g}\sin(\varphi_r - \varphi_D)\cos(\theta_D)\, sin\left(\frac{kg}{2}\left(\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\sin(\theta_r)\right)\right)\psi(x,y,z,t)}
\end{aligned}$$

1.4 SV waves

Starting from the SV wave polarization displacement vector $\vec{u}$

$$\vec{u} = \begin{bmatrix} A\cos(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ A\sin(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ A\cos(\theta_r)\,\psi(x,y,z,t) \end{bmatrix}$$

$$\begin{aligned}
u_{x'} =& \begin{bmatrix} \cos(\varphi_D)\cos(\theta_D) & \sin(\varphi_D)\cos(\theta_D) & \sin(\theta_D) \\ -\sin(\varphi_D) & \cos(\varphi_D) & 0 \\ -\sin(\theta_D)\cos(\varphi_D) & \cos(\theta_D) & \cos(\theta_D) \end{bmatrix} \begin{bmatrix} A\cos(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ A\sin(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ A\cos(\theta_r)\,\psi(x,y,z,t) \end{bmatrix} \\
=& \boxed{A(\cos(\theta_D)\sin(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\cos(\theta_r))\psi(x,y,z,t)}
\end{aligned}$$

We apply a time derivative to get the velocity vector $\dot{\vec{u}}$

$$\dot{\vec{u}} = \begin{bmatrix} Akic\cos(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ Akic\sin(\varphi_r)\sin(\theta_r)\,\psi(x,y,z,t) \\ Akic\cos(\theta_r)\,\psi(x,y,z,t) \end{bmatrix}$$

$$\dot{u}_{x'} = \frac{\delta \dot{u}}{\delta t} = \boxed{Akic(\cos(\theta_D)\sin(\theta_r)\cos(\varphi_D - \varphi_r) + \sin(\theta_D)\cos(\theta_r))\psi(x,y,z,t)}$$

The strain-rate tensor component in the ij-direction is obtained by taking the spatial gradient, giving $\dot{\epsilon}_{ij}$

$$\begin{aligned}
\dot{\epsilon}_{xx} =& \frac{-k^2cA}{2}\cos(\varphi_r)^2\sin(2\theta_r)\,\psi(x,y,z,t) & & \\
\dot{\epsilon}_{xy} =& \frac{-k^2cA}{4}\sin(2\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) & \dot{\epsilon}_{yz} =& -k^2cA\sin(\varphi_r)\sin(\theta_r)^2\,\psi(x,y,z,t) \\
\dot{\epsilon}_{xz} =& -k^2cA\cos(\varphi_r)\sin(\theta_r)^2\,\psi(x,y,z,t) & \dot{\epsilon}_{zx} =& k^2cA\cos(\varphi_r)\cos(\theta_r)^2\,\psi(x,y,z,t) \\
\dot{\epsilon}_{yx} =& \frac{-k^2cA}{4}\sin(2\varphi_r)\sin(2\theta_r)\,\psi(x,y,z,t) & \dot{\epsilon}_{zy} =& k^2cA\sin(\varphi_r)\cos(\theta_r)^2\,\psi(x,y,z,t) \\
\dot{\epsilon}_{yy} =& \frac{-k^2cA}{2}\sin(2\varphi_r)\cos(\theta_r)\,\psi(x,y,z,t) & \dot{\epsilon}_{zz} =& \frac{k^2cA}{2}\sin(2\theta_r)\,\psi(x,y,z,t)
\end{aligned}$$

Substituting in equation 2 we get the strain rate in direction xx'

$$\begin{aligned}
\dot{\epsilon}^{SV}_{xx'} =& k^2cA\left[\frac{-\cos(\varphi_D)^2\cos(\theta_D)^2\cos(\varphi_r)^2\sin(2\theta_r)}{2} - \frac{\sin(\varphi_D)\cos(\varphi_D)\cos(\theta_D)^2\sin(2\varphi_r)\sin(2\theta_r)}{2}\right. \\
&+\cos(\varphi_D)\cos(\theta_D)\sin(\theta_D)\left(-\cos(\varphi_r)\sin(\theta_r)^2+\cos(\varphi_r)\cos(\theta_r)^2\right) \\
&-\frac{\sin(\varphi_D)^2\cos(\theta_D)^2\sin(\varphi_r)^2\sin(2\theta_r)}{2}+\cos(\theta_D)\sin(\theta_D)\sin(\varphi_D)\left(-\sin(\varphi_r)\sin(\theta_r)^2+\sin(\varphi_r)\cos(\theta_r)^2\right) \\
&\left.+\frac{\sin(\theta_D)^2\sin(2\theta_r)}{2}\right]\psi(x,y,z,t) \\
=& \frac{k^2cA}{2}\left[-\cos(\theta_D)^2\sin(2\theta_r)\left(\cos(\varphi_D)^2\cos(\varphi_r)^2+\frac{\sin(2\theta 1)\sin(2\varphi_r)}{2}+\sin(\varphi_D)^2\sin(\varphi_r)^2\right)\right. \\
&\left.+\sin(2\theta_D)\cos(2\theta_D)\left(\cos(\varphi_D)\cos(\varphi_r)+\sin(\varphi_D)\sin(\varphi_r)\right)+\sin(\theta_D)^2\sin(2\theta_D)\right]\psi(x,y,z,t) \\
=& \boxed{\frac{k^2cA}{2}\left[-\cos(\theta_D)^2\sin(2\theta_r)\cos(\varphi_D-\varphi_r)^2+\sin(2\theta_D)\cos(2\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)^2\sin(2\theta_r)\right]\psi(x,y,z,t)} \\
=& \boxed{\begin{array}{l} k^2cA\left[-\cos(\theta_D)\sin(\theta_r)\cos((\varphi_D-\varphi_r))+\sin(\theta_D)\cos(\theta_r)\right] \\ \times\left[\cos(\theta_D)\cos(\theta_r)\cos((\varphi_D-\varphi_r))+\sin(\theta_D)\sin(\theta_r)\right]\psi(x,y,z,t)\end{array}}
\end{aligned}$$

To consider the gauge length effect, we average the strain over a linear segment of direction $[\cos(\varphi_D)\cos(\theta_D), \sin(\varphi_D)\cos(\theta_D), \sin(\theta_D)]$ and length g.

$$
\begin{aligned}
\dot{\epsilon}^{SV}_{xx'g} =& \frac{1}{g}\int_{\frac{+g}{2}}^{\frac{-g}{2}} \dot{\epsilon}^{SV}_{xx'}(x+v\cos(\varphi_D)\cos(\theta_D), y+v\sin(\varphi_D)\cos(\theta_D), z+v\sin(\theta_D), t)dv \\
=& \frac{k^2cA}{g}\big[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\big]\big[-\cos(\theta_D)\sin(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\cos(\theta_r)\big] \\
& \times \int_{\frac{+g}{2}}^{\frac{-g}{2}} e^{ik(ct-(x+v\cos(\varphi_D)\cos(\theta_D))\cos(\varphi_r)\cos(\theta_r)-(y+v\sin(\varphi_D)\cos(\theta_D))\sin(\varphi_r)\cos(\theta_r)-(z+v\sin(\theta_D))\sin(\theta_r)}dv \\
=& \frac{2k^2cA}{kg}\frac{\big[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\big]\big[-\cos(\theta_D)\sin(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\cos(\theta_r)\big]}{\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r-\varphi_D)+\sin(\theta_D)\sin(\theta_r)} \\
& \sin\left(\frac{kg}{2}\big(\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\big)\right)\psi(x,y,z,t) \\
=& \boxed{\begin{aligned}&\frac{2ckA}{g}\big[-\cos(\theta_D)\sin(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\cos(\theta_r)\big] \\ &\times\sin\left(\frac{kg}{2}\big(\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\big)\right)\psi(x,y,z,t)\end{aligned}}
\end{aligned}
$$

2 Looped Gauge length sensitivity equations demonstration

2.1 Horizontal loop

For a horizontaly looped gauge length, the dip ange θ_D is zero. To compute the averaged strain rate for each wave polarization over the loop, we evaluate:

$$\frac{1}{g}\int \dot{\epsilon}_{xx'}\,ds$$

$$\frac{1}{2\pi R}\oint \dot{\epsilon}_{xx'}\,Rd\varphi_D$$

Since the gauge length is a circle, its length can be rewritten as $g = 2\pi R$, where R is the loop radius. We assume here that the wavelength is much larger than the loop size. Hence, the oscillatory term can be considered as constant over the loop.

2.1.1 P Waves

Starting with the point strain-rate sensitivity of a linear DAS segment to P-waves

$$\dot{\epsilon}^{p}_{xx'} = k^2cA\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\right]^2\psi(x,y,z,t)$$

Since $\theta_D = 0$, the equation can be simplified:

$$\dot{\epsilon}^{p}_{xx'} = k^2cA\big[\cos(\theta_r)\cos(\varphi_D-\varphi_r)\big]^2\psi(x,y,z,t)$$

Averaging over the loop:

$$
\begin{aligned}
\dot{\epsilon}^{P}_{hl} =& \frac{k^2cA}{2\pi R}\cos(\theta_r)^2\psi(x,y,z,t)\oint\cos(\varphi_D-\varphi_r)^2\,Rd\varphi_D \\
=& \frac{k^2cA}{2\pi}\cos(\theta_r)^2\psi(x,y,z,t)\oint(\frac{1}{2}+\frac{1}{2}\cos(2(\varphi_D-\varphi_r)))d\varphi_D
\end{aligned}
$$

Evaluating the integral:

$$\oint\cos(2(\varphi_D-\varphi_r))\,d\varphi_D = 0 \quad \text{and} \quad \oint 1\,d\varphi_D = 2\pi$$

Therefore:

$$\boxed{\dot{\epsilon}_{hl}^{P} = \frac{k^2 cA}{2} \cos(\theta_r)^2 \, \psi(x,y,z,t)}$$

2.1.2 SH Waves

From the point strain-rate sensitivity of a linear DAS segment, the strain rate due to SH-waves is given by:

$$\dot{\epsilon}_{xx'}^{SH} = ck^2 A \sin(\varphi_r - \varphi_D)\cos(\theta_D)\left[\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r - \varphi_D) + \sin(\theta_D)\sin(\theta_r)\right]\psi(x,y,z,t)$$

Since $\theta_D = 0$, it simplifies to:

$$\dot{\epsilon}_{xx'}^{SH} = \frac{k^2 cA}{2} \cos(\theta_r)\sin(2(\varphi_r - \varphi_D))\ \psi(x,y,z,t)$$

The loop-averaged strain rate is given by:

$$\dot{\epsilon}_{hl}^{SH} = \frac{k^2 cA}{2\pi R} R\cos(\theta_r) \oint \frac{1}{2}\sin(2(\varphi_r - \varphi_D))\, d\varphi_D \, \psi(x,y,z,t)$$

Evaluating the integral:

$$\oint \sin(2(\varphi_r - \varphi_D))\, d\varphi_D = 0$$

Therefore, the loop-averaged SH-wave strain rate is:

$$\boxed{\dot{\epsilon}_{hl}^{SH} = 0}$$

2.1.3 SV Waves

From the point strain-rate sensitivity of a linear DAS segment, the strain rate due to SV-waves is given by:

$$\begin{aligned}\dot{\epsilon}_{xx'}^{SV} =& k^2 cA\left[-\cos(\theta_D)\sin(\theta_r)\cos((\varphi_D - \varphi_r)) + \sin(\theta_D)\cos(\theta_r)\right] \\ &\times \left[\cos(\theta_D)\cos(\theta_r)\cos((\varphi_D - \varphi_r)) + \sin(\theta_D)\sin(\theta_r)\right]\psi(x,y,z,t)\end{aligned}$$

Since $\theta_D = 0$:

$$\begin{aligned}\dot{\epsilon}_{xx'}^{SV} =& k^2 cA\Big[-\sin(\theta_r)\cos(\varphi_D - \varphi_r)\Big]\Big[\cos(\theta_r)\cos(\varphi_D - \varphi_r)\Big]\psi(x,y,z,t) \\ =& -\frac{k^2 cA}{2}\sin(2\theta_r)\left(\frac{1}{2} + \frac{1}{2}\cos(2(\varphi_D - \varphi_r))\right)\end{aligned}$$

we write the loop-averaged strain rate as:

$$\dot{\epsilon}_{hl}^{SV} = -\frac{k^2 cA}{4\pi R} R\sin(2\theta_r) \oint \left(\frac{1}{2} + \frac{1}{2}\cos(2(\varphi_D - \varphi_r))\right) d\varphi_D$$

Evaluating the integral:

$$\oint \left(\frac{1}{2} + \frac{1}{2}\cos(2(\varphi_D - \varphi_r))\right) d\varphi_D = \pi$$

Therefore:

$$\boxed{\dot{\epsilon}_{hl}^{SV} = -\frac{k^2 cA}{4}\sin(2\theta_r)\,\psi(x,y,z,t)}$$

2.2 Vertical Loop

For a vertical loop contained in a plane with orientation φ_D, we evaluate the loop-averaged strain rate over all dip θ_D:

$$\frac{1}{g}\int \dot{\epsilon}_{xx'}\, ds$$

$$\frac{1}{2\pi R}\oint \dot{\epsilon}_{xx'}\, R d\theta_D$$

Since the gauge length forms a circle, it can be written as $g = 2\pi R$, with R being the loop's radius. We assume a wavelengths much larger than the loop size. Hence, the oscillatory term can be considered as constant over the loop.

2.2.1 P Waves

From the point strain-rate sensitivity of a linear DAS segment, the strain rate due to P-waves is given by:

$$\dot{\epsilon}^{p}_{xx'} = k^2 cA\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\right]^2\psi(x,y,z,t)$$

Averaging over the vertical loop yields:

$$\begin{aligned}\dot{\epsilon}^{P}_{vl} =& \frac{k^2cA}{2\pi R}R\oint\left[\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)+\sin(\theta_D)\sin(\theta_r)\right]^2 d\theta_D\,\psi(x,y,z,t)\\ =& \frac{k^2cA}{2\pi}\oint\left[\cos(\theta_D)^2\cos(\theta_r)^2\cos(\varphi_D-\varphi_r)^2+2\cos(\theta_D)\cos(\theta_r)\cos(\varphi_D-\varphi_r)\sin(\theta_D)\sin(\theta_r)\right.\\ &\left.+\sin(\theta_D)^2\sin(\theta_r)^2\right]d\theta_D\,\psi(x,y,z,t)\end{aligned}$$

We evaluate the three parts of the integral separately:
- First term:

$$\cos(\theta_r)^2\cos(\varphi_D-\varphi_r)^2\oint\cos(\theta_D)^2\,d\theta_D = \pi\cos(\theta_r)^2\cos(\varphi_D-\varphi_r)^2$$

- Second term:

$$2\cos(\theta_r)\cos(\varphi_D-\varphi_r)\sin(\theta_r)\oint\cos(\theta_D)\sin(\theta_D)\;d\theta_D = 0$$

- Third term:

$$\sin(\theta_r)^2\oint\sin(\theta_D)^2\;d\theta_D = \quad \pi\sin(\theta_r)^2$$

Substituting:

$$\boxed{\dot{\epsilon}^{P}_{vl} = \frac{k^2cA}{2}(\cos(\theta_r)^2\cos(\varphi_D-\varphi_r)^2+S^2_{\theta_r})\,\psi(x,y,z,t)}$$

2.2.2 SH waves

From the point strain-rate sensitivity of a linear DAS segment, the strain rate due to SH-waves is given by:

$$\dot{\epsilon}^{SH}_{xx'} = ck^2A\sin(\varphi_r-\varphi_D)\cos(\theta_D)\left[\cos(\theta_r)\cos(\theta_D)\cos(\varphi_r-\varphi_D)+\sin(\theta_D)\sin(\theta_r)\right]\psi(x,y,z,t)$$

Averaging over the vertical loop yields:

$$\dot{\epsilon}^{SH}_{vl} = \frac{k^2cA}{2\pi R}R\oint\cos(\theta_D)^2\cos(\theta_r)\,\frac{\sin(2(\varphi_D-\varphi_r))}{2}+\sin(\varphi_r-\varphi_D)\frac{\sin(2\theta_D)}{2}\sin(\theta_r)\;d\theta_D\,\psi(x,y,z,t)$$

We evaluate the two parts of the integral separately:
- First term:

$$\oint \cos(\theta_D)^2 \, \cos(\theta_r) \, \frac{\sin(2(\varphi_D - \varphi_r))}{2} \, d\theta_D = \pi \, \cos(\theta_r) \, \frac{\sin(2(\varphi_D - \varphi_r))}{2}$$

- Second term:

$$\oint \sin(\varphi_r - \varphi_D) \, \frac{\sin(2\theta_D)}{2} \sin(\theta_r) \, d\theta_D = 0$$

Substituting into the averaged expression:

$$\boxed{\dot{\epsilon}_{vl}^{SH} = \frac{k^2 cA}{4} \, \cos(\theta_r) \, \sin(2(\varphi_D - \varphi_r)) \, \psi(x, y, z, t)}$$

2.2.3 SV waves

From the point strain-rate sensitivity for a linear DAS segment, the strain rate due to SV-waves is given by:

$$\dot{\epsilon}_{xx'}^{SV} = \frac{k^2 cA}{2} \left[-\cos(\theta_D)^2 \sin(2\theta_r) \cos(\varphi_D - \varphi_r)^2 + \sin(2\theta_D) \cos(2\theta_r) \cos(\varphi_D - \varphi_r) + \sin(\theta_D)^2 \sin(2\theta_r) \right] \psi(x, y, z, t)$$

Averaging over the vertical loop:

$$\begin{aligned} \dot{\epsilon}_{vl}^{SV} = & \frac{k^2 cA}{4\pi R} R \oint \Big[-\cos(\theta_D)^2 \sin(2\theta_r) \cos(\varphi_D - \varphi_r)^2 + \sin(2\theta_D) \cos(2\theta_r) \cos(\varphi_D - \varphi_r) \\ & + \sin(\theta_D)^2 \sin(2\theta_r) \Big] \, d\theta_D \, \psi(x, y, z, t) \end{aligned}$$

We evaluate the three parts of the integral separately:
- First term:

$$\oint -\cos(\theta_D)^2 \sin(2\theta_r) \cos(\varphi_D - \varphi r)^2 \, d\theta_D = -\pi \sin(2\theta_r) \cos(\varphi_D - \varphi r)^2$$

- Second term:

$$\oint \sin(2\theta_D) \cos(2\theta_r) \cos(\varphi_D - \varphi_r) \, d\theta_D = 0$$

- Third term:

$$\oint \sin(\theta_D)^2 \sin(2\theta_r) \, d\theta_D = \pi \sin(2\theta_r)$$

Substituting:

$$\boxed{\dot{\epsilon}_{vl}^{SV} = \frac{k^2 cA}{4} \sin(2\theta_r) \sin(\varphi_D - \varphi_r)^2 \, \psi(x, y, z, t)}$$

3 Ray tracing

To assess the accuracy of the ray-tracing approach and determine appropriate numerical parameters for Pykonal, we perform a synthetic sensitivity analysis. This framework allows us to systematically evaluate how the spatial discretization of the velocity model affects ray stability, computational cost, and the intrinsic uncertainty associated with the numerical solution.

3.1 Experimental Setup

We generate an ensemble of 10,000 randomly distributed source-receiver pairs within a 1 km^3 velocity model. The model consists of a checkerboard pattern composed of cubic heterogeneities, with cube dimensions and velocity contrasts summarized in Table S1 and represented by the vertical lines in Figure S2 A and C. To

Cube size (m)	5	15	20	25	30	35	40	45	100
Velocity (m/s)	100	100	100	100	100	100	100	100	1000
Velocity delta (Δv)	100	100	100	100	100	100	100	100	100

Table S1: Parameters of the checkerboard velocity model.

satisfy the high-frequency assumption underlying the Eikonal equation, we apply a spatial low-pass filter characterized by an attenuation of 0.99 at a wavelength of 30 m. The filtering is implemented through convolution with a Gaussian kernel, whose standard deviation (σ) is defined by

$$\sigma = \frac{\lambda}{\pi}\sqrt{-2\ln(0.99)}\,. \tag{4}$$

The corresponding spatial attenuation response is shown in Figure S2 C as a function of element size (λ). For each source-receiver pair, we compute ray paths while progressively varying the spatial resolution of the velocity grid. In Pykonal, the Euler step used to compute the travel-time gradient and trace the ray path is a quarter to the grid's node spacing. Therefore, changing the grid resolution provides a direct measure of the sensitivity of the resulting ray paths to numerical discretization. Ray stability is quantified in two ways. First, we measure the angular deviation at the source and receiver relative to the ray obtained using the finest spatial resolution. Second, we evaluate the global difference between ray trajectories by calculating the Euclidean distance between corresponding nodes along the paths.

3.2 Results

The median ray deviation across the 10,000 source-receiver pairs remains stable for grid spacings up to approximately half of the characteristic filtering wavelength (Figure S2 A), corresponding to a step size of 15 m or a node spacing of 60 m. Beyond this limit, increasing the grid spacing progressively modifies the ray geometry, with both the incidence angles and the average Euclidean distance between corresponding ray nodes increasing monotonically. To quantify the intrinsic uncertainty associated with the numerical discretization, we restrict the analysis to converged solutions (ray step size <15 m). For each source-receiver pair, we calculate the standard deviation of the source and receiver incidence angles across the different grid resolutions and examine their distribution over the 10,000 realizations. The resulting distributions (Figure S2D-E) peak at 0.59° for the source angle and 3.30° for the receiver angle. The receiver distribution (Figure S2 E) also exhibits a pronounced long tail, which we attribute to cases where multiple ray paths are possible and travel-time discretizations affects path selection. Finally, we evaluate the computational implications of increasing the spatial resolution. The runtime increases rapidly with decreasing grid spacing, following an approximately cubic scaling relationship (Figure S2 B). This trade-off is particularly important for DAS applications, where ray paths must be independently computed for a large number of channels.

These results provide a first-order estimate of ray-tracing stability and uncertainty. The synthetic model assumes uniform heterogeneity amplitudes and does not explicitly account for the influence of ray length, both of which may affect the magnitude of numerical errors in more realistic velocity structures.

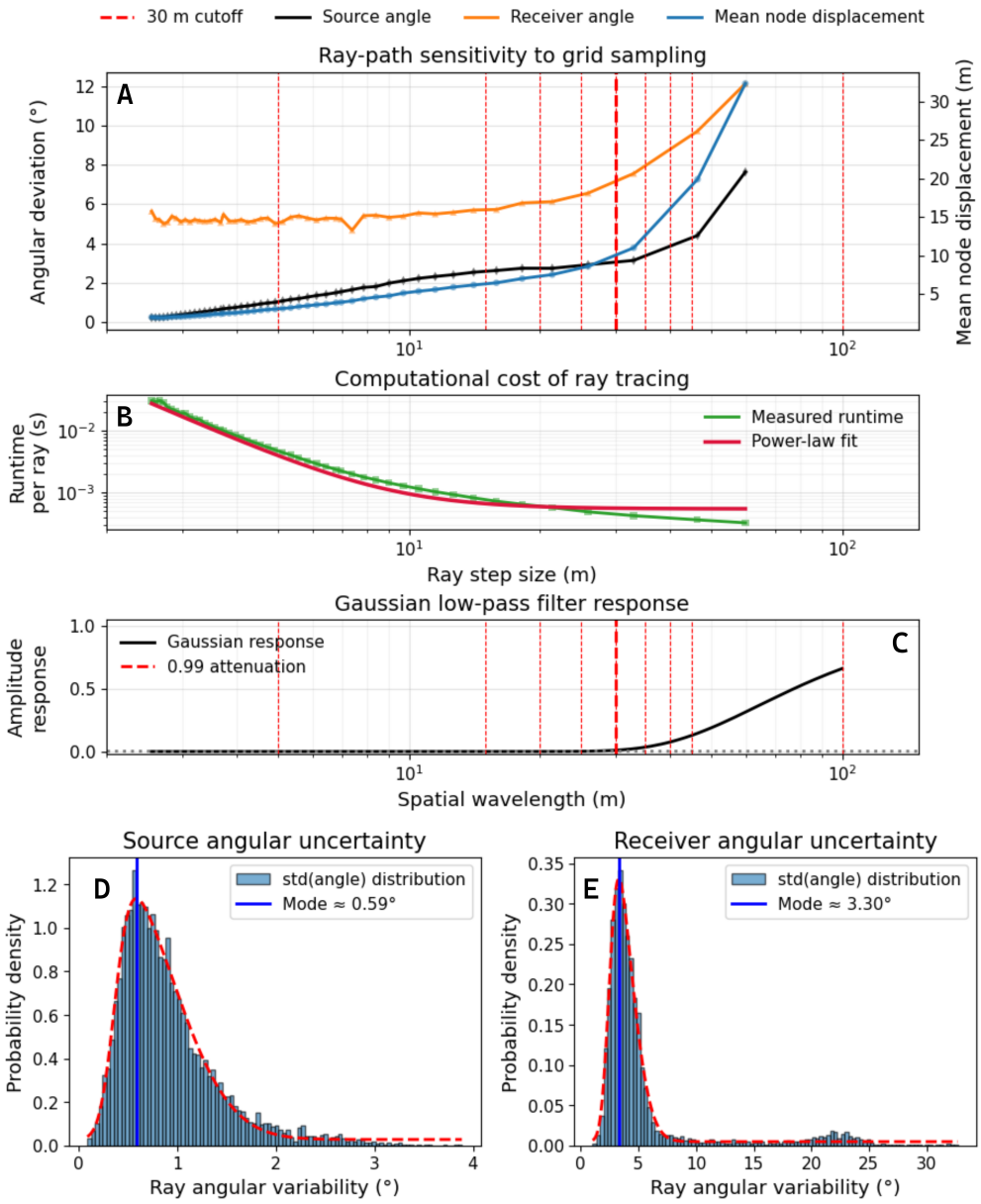


Figure S2: Ray-tracing stability and uncertainty as a function of velocity-grid resolution. (A) Median angular deviation at the source (black) and receiver (orange), and mean Euclidean displacement between corresponding ray nodes (blue) relative to the highest-resolution solution. Values are computed over an ensemble of 10,000 randomly distributed source–receiver pairs within a 1 km^3 checkerboard velocity model (Table S1). (B) Average computational cost per ray-tracing operation as a function of grid spacing, together with the best-fitting power-law relationship ($run\ time = 0.4 * step\ size^{-3}. + 5.5e-04$). (C) Spatial response of the Gaussian low-pass filter used to satisfy the high-frequency approximation of the Eikonal equation. The filter corresponds to a transmission coefficient of 0.99 at a wavelength of 30 m. Vertical dashed lines in panels A and C indicate the characteristic length scales of the velocity heterogeneities. (D–E) Distribution of angular uncertainty caused by grid discretization at the source and receiver, respectively. The uncertainty is estimated for each source–receiver pair from the standard deviation of ray angles obtained using grid spacings below 15 m. The distributions peak at 0.59° for the source angle and 3.3° for the receiver angle. The broader receiver distribution and non-zero baseline reflect the influence of multiple possible ray paths and numerical path selection.